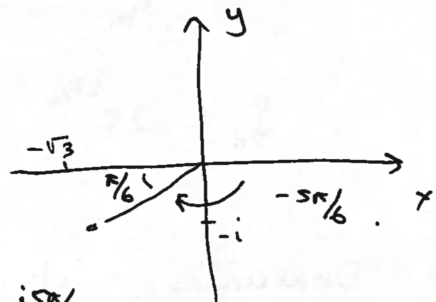
Test # 1 - Solutions

1. (a) We have $z = -2\sqrt{3} - 2i = 2(-\sqrt{3} - i)$

$|z| = |2(-\sqrt{3}-i)| = 2\sqrt{3+1} = 4$

$\text{Arg}(z) = -\frac{5\pi}{6}$



The polar form of z is

$$z = 4\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(\frac{-5\pi}{6}\right)\right) = 4e^{-i5\pi/6}$$

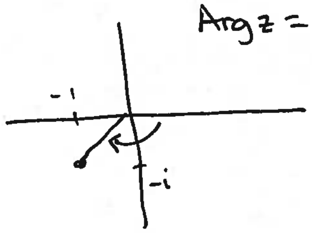
$$= 4\left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right)$$

(b) Evaluate $(-1-i)^9 = z^9$

First we convert z to exponential form

$|z| = |-1-i| = \sqrt{2} = r$

$\text{Arg}\, z = -\frac{3\pi}{4} = \theta$



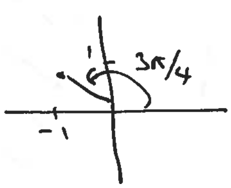
$$z^n = r^n e^{in\theta}$$

$$z^9 = \left(2^{1/2} e^{-i\frac{3\pi}{4}}\right)^9$$

$$= 2^{9/2} e^{-i\frac{27\pi}{4}}$$

now $\frac{27\pi}{4} = \frac{24\pi + 3\pi}{4} = \underbrace{6\pi}_{3\cdot 2\pi} + \frac{3\pi}{4}$

so $= 2^{9/2} e^{-i\frac{3\pi}{4}}$

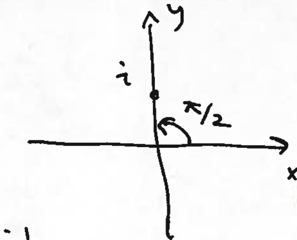


$$= 2^{9/2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$$

$$= 2^{9/2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$= 2^4(-1-i)$$

(c) Consider $(8i)^{1/3} = 8^{1/3}\left(e^{i\pi/2}\right)^{1/3}$



$$= 2\left(e^{i\left(\frac{\pi}{2}+2k\pi\right)}\right)^{1/3} \quad \text{by periodicity}$$

there are 3 distinct roots

$$\rho_k = 2e^{i\frac{\pi+4k\pi}{6}} \quad \text{for } k=0,1,2$$

$$\rho_0 = 2e^{i\pi/6}, \quad \rho_1 = 2e^{i5\pi/6}, \quad \rho_2 = 2e^{i9\pi/6} = 2e^{-i3\pi/6} = 2e^{-i\pi/2}$$

(d) Evaluate $\sin\frac{\pi i}{2}$.

by definition $\quad \sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$

so $\quad \sin\left(\dfrac{\pi i}{2}\right) = \dfrac{e^{i\left(\frac{\pi i}{2}\right)}-e^{-i\left(\frac{\pi i}{2}\right)}}{2i} = \dfrac{i}{i}\,\dfrac{e^{-\frac{\pi}{2}}-e^{\frac{\pi}{2}}}{2i}$

$$= i\,\frac{e^{\pi/2}-e^{-\pi/2}}{2} = i\sinh\frac{\pi}{2}$$

2. (a) We have $(\operatorname{Re} z)^2 > 1$

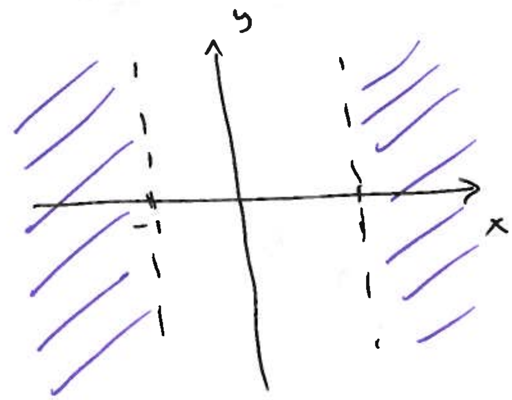
set $z = x+iy$

$\operatorname{Re} z = x$ $\qquad$ so we need $x^2 > 1$

$(\operatorname{Re}(z))^2 = x^2$

there are no restriction on y, but we need $x>1$ and $x<-1$

This is open, but not connected so not a domain



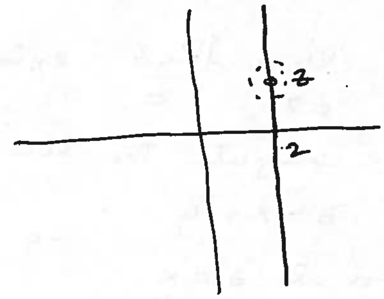
2 (b) Consider $\mathrm{Re}(\bar{z}-i)=2$

set $z=x+iy$ then $\mathrm{Re}(\bar{z}-i) = \mathrm{Re}(x-iy-i)$

$$= x$$

The set $x=2$ correspond to the vertical line at $x=2$.

This is not open since every neighborhood of $z \in \{x+iy \mid x=2\}$ contains points not on the line / so not a domain



Note that it is connected.

3. f is continuous at $z=i$ if $\lim_{z\to i} \frac{iz}{3} = f\left(\frac{iz}{3}\right) = -\frac{1}{3}$.

Fix $\varepsilon>0$, we need to show that $\left|f(z)-\left(-\frac{1}{3}\right)\right|<\varepsilon$

whenever $|z-i|<\delta$.

Consider $\left|f(z)-\left(-\frac{1}{3}\right)\right| = \left|\frac{iz}{3}-\left(-\frac{1}{3}\right)\right|$

$$= \frac{1}{3}|iz+1|$$

$$= \frac{1}{3}|i(z-i)|$$

$$= \frac{1}{3}|z-i| < \frac{1}{3}\delta \quad \text{by assumption}$$

so choose $\delta=\varepsilon$ then

$$\left|f(z)-\left(-\frac{1}{3}\right)\right| < \frac{1}{3}\delta = \frac{1}{3}\varepsilon < \varepsilon$$

∎

4. Consider $f(z) = \begin{cases} 0 & z=0 \\ \dfrac{\operatorname{Im} z}{z} & z \neq 0 \end{cases}$.

$f(z)$ is continuous at $z=0$ provided the

$\lim_{z\to 0} \dfrac{\operatorname{Im} z}{z}$ exist and equal 0.

First, we compute the limit

set $z = x+iy$ so $\lim_{z\to 0} \dfrac{\operatorname{Im} z}{z} = \lim_{(x,y)\to(0,0)} \dfrac{y}{x+iy}$

then $\operatorname{Im} z = y$

Claim The limit doesn't exist

(1) Take limit as $z\to 0$ along the real axis $(y=0,\ x\to 0)$

$$\lim_{z\to 0} \frac{\operatorname{Im} z}{z} = \lim_{(x,0)\to(0,0)} \frac{0}{x} = 0$$

(2) Take limit as $z\to 0$ along the y axis $(x=0,\ y\to 0)$

$$\lim_{z\to 0} \frac{\operatorname{Im} z}{z} = \lim_{(0,y)\to(0,0)} \frac{y}{iy} = \frac{1}{i} = -i$$

Since the limit along two different paths give different answers, the limit doesn't exist

therefore the function is not continuous at $z=0$

5. $f(z)$ is analytic in a set S if f is differentiable for every $z \in S$, and there exists a neighborhood of z such that f is differentiable at every point in the neighborhood.

First, we use the Cauchy-Riemann equation to see where $f(z) = z\bar{z}$ is differentiable

C.-R. Eq.:

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}}$$

set $f(z) = u(x,y) + i\,v(x,y)$

$$f(z) = z\bar{z} = |z|^2 = x^2 + y^2$$

so $u(x,y) = x^2 + y^2$
$v(x,y) = 0$

$u_x = 2x$, $u_y = 2y$, $v_x = 0$, and $v_y = 0$

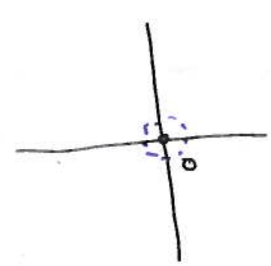
u & v and all partial derivatives are continuous on $\mathbb{C}$ (polynomials) by our Theorem, the function will be differentiable wherever the Cauchy-Riemann equations are satisfied

we need
$$u_x = v_y \Rightarrow 2x = 0$$
$$u_y = -v_x \Rightarrow 0 = -2y$$

this is true only if $x = y = 0$ or $z = 0$

So $f(z)$ is differentiable at $z = 0$.

However, $z\bar{z}$ is nowhere analytic because any neighborhood of $z=0$ will contain points where the function is not differentiable



6. Consider $v(x,y) = \sin x \cosh y$

(a) Show that $v(x,y)$ is harmonic

Def^{n} A real-valued function $\phi(x,y)$ with continuous 2nd order partial derivatives in a domain D that satisfies $\nabla^2 \phi = 0$ is called harmonic.

$$\frac{\partial v}{\partial x} = \cos x \cosh y \qquad \frac{\partial v}{\partial y} = \sin x \sinh y$$

$\sin x$, $\cos x$, $\sinh y$, and $\cosh y$ are all continuous.

and

$$\frac{\partial^2 v}{\partial x^2} = -\sin x \cosh y \qquad \frac{\partial^2 v}{\partial y^2} = \sin x \cosh y$$

$$\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y} = \cos x \sinh y.$$ are also continuous.

By Clairaut since 1st order partial are continuous

Moreover $\nabla^2 v = v_{xx} + v_{yy} = -\sin x \cosh y + \sin x \cosh y = 0$

so we conclude that $v(x,y)$ is harmonic in the plane.

(b) We use the fact that given an harmonic function in an open disc we can always find another harmonic function so that

$$f(z) = u(x,y) + i\,v(x,y) \text{ is analytic.}$$

u & v are harmonic conjugates.

Note that for f to be analytic, it is necessary that u & v satisfy the Cauchy Riemann Equations

We have $v(x,y) = \sin x \cosh y$

(1) $u_x = v_y \Rightarrow u(x,y) = \int v_y \, dx$

$$= \int \sin x \sinh y \, dx$$

$$= -\cos x \sinh y + C(y)$$

where $C(y)$ is a function that only depends on y. possibly a constant.

(2) $u_y = -v_x \Rightarrow -\cos x \cosh y + C'(y) = -\cos x \cosh y$

so $C'(y) = 0 \Rightarrow C(y) = C$ a constant

therefore $u(x,y) = -\cos x \sinh y + C$

Since $f(0) = (1,0) = 1 + i0$

we need $u(0,0) = 1 = 0 + C \Rightarrow C = 1$

since $\sinh 0 = 0$

so $u(x,y) = -\cos x \sinh y + 1$

And $f(z) = 1 - \cos x \sinh y + i \sin x \cosh y$

is the desired function.